Strict Logical Notation Is Not a Part of the Problem but a Part of the Solution for Teaching High-School Mathematics

M. Peltomäki and T. Salakoski
Department of Information Technology
University of Turku and Turku Centre for Computer Science
Lemminkäisenkatu 14-18A, 20520 Turku

mia.peltomaki@utu.fi

Abstract

During the years 2001-2004 we have conducted an experiment at a Finnish high school, where we taught advanced mathematics to students interested in computer science using a new teaching method based on structured derivations and their computer-assisted use. With the new method, we aim at a deeper abstract logical thinking via more exact and perceptible expression. In this paper, we describe the experiment and present the preliminary results. The quantitative learning results show that in general, the new method is definitely not worse than the traditional way and for talented students it yields substantially better results. Furthermore, the method clearly has the potential of improving the quality of learning.

1 Introduction

At high school and lower level education, using exact formalism in definitions and proofs is often avoided even in mathematics courses and thus the design of proofs is not taught either. Students do see a few proofs and may even be asked to develop a few themselves, but usually there is only a little or no discussion at all about the principles or strategies for designing proofs. The concept of mathematical proof, or computational derivation, is not made perceptible to the student and, consequently, students ability of logical reasoning and abstract thinking remain poor. It seems that high school does not give students the sufficient thinking tools for employing mathematics in solving practical problems, a skill especially needed at university level computer science education.

After nine year-long basic education in Finland, students aged 15 or 16 continue mainly either in general upper secondary school or in vocational upper secondary education and training. In this text, we use the phrase high school for Finnish general upper secondary school, which is public and free for all students. The teaching is given at courses consisting of about 30 classroom hours. After completing at least 75 courses students aged 18 or 19 participate in the national matriculation examinations.

According to the current high school level curriculum, computer science is no more mentioned as an independent subject. The aims of mathematics studies include among other things that students appreciate the exactness and clarity in presentations. The new curriculum to be adopted in 2005 emphasizes the evaluation of students skills focusing to the selection of an appropriate method, and the justification of exact conclusions. Learning to see the mathematical information as a logical structure has been added to the teaching aims. (Opetushallitus, 1994, 2003)

Despite the noble objectives, logical notation is in practise very seldom used at high school. Logic, quantifiers, and the training of proving principles come along with the new curriculum, but only on the optional advanced courses following the compulsory courses. We believe, however, that it would be better to explicitly use logic already on the compulsory courses – but as a tool rather than an object of study.

Starting with logic makes the course seem coherent and provides students with a supportive framework, which they can lean on while the various aspects of the proof and counterexample are falling into place. It builds students confidence in the rationality of the mathematical enterprise and helps allay their fear of failure. Determining truth and falsity of mathematical statements is so complex, that even when they are motivated, students often fail to really "get it" if they do not have any prior experience with basic logical tools. (Epp, 2003)

In this paper, we show how structured derivation, which is an extension on the calculational proof style written with logical notation, could be used in high-school mathematics to facilitate the students to write the justifications of the solutions. Our objectives are the following. Firstly, we want to promote exact expression by employing formal notation. Secondly, we try to make mathematical thinking and logical reasoning visible and hence more perceptible. Finally, we wish to deepen the students understanding of logical abstractions and the processes of thought by making use of computer-assisted manipulation of structural derivations.

The paper is organized as follows. In Chapter 2, we very shortly introduce structured derivations. Our teaching experiment at is described in Chapter 3, and in Chapter 4, we present preliminary results. We end the paper by discussion and conclusions.

2 Structured Derivations, briefly

Dijkstra and Scholten (1990) introduce the calculational proof format in their book *Predicate Calculus and Program Semantics*. They begin by making the observation that a great many proofs can be described as a series of transformations. Inspired by the clarity and the readability of the format, calculational paradigm for manipulating mathematical expressions emerged. According to the paradigm, mathematical expressions are transformed step by step from the initial expression to a solution. Each new version of the expression is written on a new line and between the two lines is written a symbol denoting the relationship between the expressions together with a justification for the validity of the step. The paradigm has been attributed to W. Feijen, and described in detail by van Gasteren (1990).

As an example of applying the method on the courses in detail, the derivation of solving an inequality is shown both in its fully expanded form as written on the whiteboard and in the form with hidden sub-derivations (Example 1).

Example 1.
Solving the inequality $|x\text{-}4| \geq 3x$ shown
(a) in the form with hidden sub-derivations and
(b) in its fully expanded form as written on the board

(a)
   $|x-4| \geq 3x$
$\equiv$ {property of absolute values}
   $(x-4 \geq 3x \vee x-4 \leq -3x)$
$\equiv$ {solve the inequalities}
… $x \leq -2 \vee x \leq 1$
$\equiv$ {simplify}
   $x \leq 1$

(b)
   $|x-4| \geq 3x$
$\equiv$ {property of absolute values}
   $(x-4 \geq 3x \vee x-4 \leq -3x)$
$\equiv$ {<u>solve the inequalities</u>}
- $x-4 \geq 3x$
  $\equiv$ {add 4 to both sides of the inequality}
  $x \geq 3x+4$
  $\equiv$ {subtract $3x$ from both sides of the inequality}
  $-2x \geq 4$
  $\equiv$ {divide both sides with -2}
  $x \leq -2$
- $x-4 \leq -3x$
  $\equiv$ {add 4 to both sides of the inequality}
  $x \leq -3x+4$
  $\equiv$ {add $3x$ to both sides of the inequality}
  $4x \leq +4$
  $\equiv$ {divide both sides with 4}
  $x \leq 1$

… $x \leq -2 \vee x \leq 1$
$\equiv$ {simplify}
   $x \leq 1$

Although structured derivations are based on the calculational proof paradigm and they are developed originally for the formal refinement of computer programs and reasoning about their correctness (Back and von Wright, 1998) this method can be used in high-school mathematics as a way of writing solutions to typical problems (Back and von Wright, 1999; Back et al., 2002). Furthermore, when solutions are very long, sub-derivations can be hidden and replaced with a link giving more detailed view of the partial solution.

3 Teaching experiment

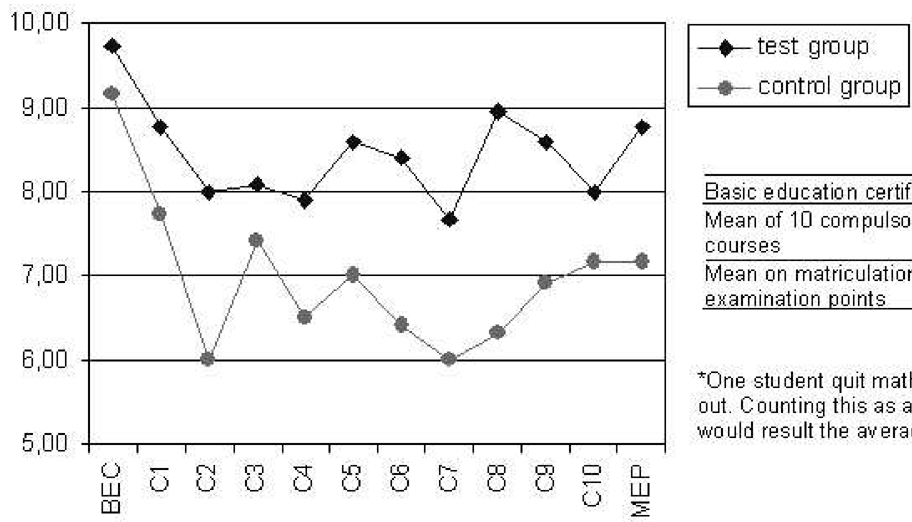
The method has been studied and tested at a Finnish high school between August 2001 and May 2004 (Kavander et al., 2001; Back et al., 2002, 2003). Every year before beginning the studies, the new students in advanced mathematics answered to an attitude enquiry. They also took an exam, which tells us about their mathematical skills. The students were then divided into three groups: the test group, the control group and the rest. The test group was selected so that most students expressing their interest in computer science were included, because we believe that the new method would turn out especially useful later when studying computer science at university level. Nevertheless, we tried to divide students to the test and control groups so that they were about on the same entry level according to the basic education certificate and the mathematical skills tested in our exam.

We used the structured derivations method when teaching new basic theory both on the whiteboard and with a computer. When using a computer and a browser, sub-derivations, consisting of a detailed solution to a part of the problem can be hidden with a link or shown depending, on how detailed a solution the students want to see for instance when checking their homework.

The results of the test group have been compared with those of the control group throughout the three-year period by arranging common examinations for each course. Finally, after finishing high school, we compared the results of the matriculation examination between the students in the test group, the control group, and all students in Finland participating in the national examination.

4 Results

In Figure 1, we present a summary of the preliminary results. We consider the development of the grades from the basic education certicate, through the ten compulsory courses of advanced mathematics, to the matriculation examination results.



	test	control
Basic education certificate (BEC)	9,73	9,17
Mean of 10 compulsory courses (C1-C10)	8,30	6,75
Mean on matriculation examination points (MEP)	8,77*	7,17

*One student quit mathematics and dropped out. Counting this as a failure (zero points) would result the average 8,52.

Figure 1: The grades results. The means of the mathematics grades in the basic education certificate (first in left); the means of the course grades of the ten compulsory courses (in the middle); and the means of the matriculation examination points (last in right) in the test and control groups in a scale from 5 to 10.

When comprehensive school students enter high school, it is customary that the grades fall. However, it seems that the group participating in the teaching experiment unconsciously received a slightly stricter course assessment than usually; most likely due to the high entry level. It can be seen that the mark 10 typically falls down to about 8.5 (Suikki, 2004) whereas in the test group, it fell down to 8.2. Consequently, the results of the matriculation examination were better than expected based on the course assessments. The accepted course grades range from 5 to 10 and here, we have scaled the grades in the matriculation examination respectively.

A more detailed analysis of the performance of the test group in the national matriculation examination is still going on. Preliminary results suggest, however, that the test group succeeded remarkably well. More than 70% of the test group got one of the two highest degrees (here, 9 or 10) while on the national level this is normalized to about 20% (in spring 2004, 21%). Detailed results are not shown here.

5 Discussion and conclusions

By using the logical notation and the method of structured derivations, as suggested in this paper, logical reasoning and abstract thinking can be made more perceptiple to the students. The logical structure of the stepwise derivation from the original formulation of the problem to the final solution, is maintained while the level of abstraction can be adjusted depending on the focus of consideration.

The quantitative learning results of the teaching experiment show that in general, the use of logical notation and structured derivations in high-school mathematics is certainly not less productive than the traditional way and, moreover, the new method seems to yield substantially better learning results for talented students. In addition, although a thorough qualitative analysis of the solutions by the test group students is still under way, the new method seems to improve the quality of learning, resulting in deeper understanding of logical abstractions and advanced mathematical thinking.

Based on our experience, most of the students are deeply committed to the style they have learned before high school. The standard textbooks used on high-school mathematics courses do not include logic. As the teaching progressed with the experimental method using structured derivations and not according to the textbook, some students got mixed and consequently, their results may have suffered. Obviously, this problem of inconsistent learning material could be avoided, if a suitable textbook existed to support the use of the new method.

Gries and Schneider (1993) have used the calculational proof style in their book *A Logical Approach to Discrete Math*, and their experiences on teaching in this way at the university level have been positive (Gries and Schneider, 1995). Our conclusion at the high-school level is similar: talented students learned easily the basic logical tools and also, they seem to make better use of the new formalism, whereas some of the less-talented students got confused with the new and traditional notation and thus their solutions to the exercises were not always logically firm. Therefore, we think that at the high-school level, the notation should be somewhat lighter than at the university level. During this three years experiment we have already revised the notation accordingly, and placed a section about the basics of logic into the beginning of the first course.

We look forward to preparing teaching material for the courses, perhaps in the form of a textbook. In addition, we aim at developing tools for writing structured derivations quickly and easily with a computer. These tools would enhance the usability of the method from the students as well as from the teachers point of view, and they could be accompanied by various guidance, verification, and assessment features. Furthermore, we are currently revising the method for incorporating a systematic way of using pictures and naming conventions for problem solving, such as in experimental settings, geometrical illustrations and data structures.

References

Back, R., Kavander, T., Nylund, M., Peltomäki, M., Salakoski, T., von Wright, J., 2002. Matemaattinen todistaminen ja abstrahointi lukion matematiikan opetuksessa – opetuskokeilu rakenteisten johtojen avulla. (Mathematical Proof and Abstraction in High-School Mathematics Education – Teaching Experiment Using Structured Derivations). In: Meisalo, V. (Ed.), Aineenopettajankoulutuksen vaihtoehdot ja tutkimus. Ainedidaktiikan symposiumi 1.2.2002. Tutkimuksia 241. Department of Teacher Education, University of Helsinki, pp. 691–697.

Back, R., Peltomäki, M., Salakoski, T., von Wright, J., 2003. Structured Derivations Supporting High-School Mathematics. In: Proceedings of 20th Annual Symposium of the Finnish Mathematics and Science Education Research Association. Department of Teacher Education, University of Helsinki, to appear.

Back, R., von Wright, J., 1998. Refinement Calculus: A Systematic Introduction. Springer-Verlag, New York.

Back, R., von Wright, J., Mar. 1999. Structured Derivations: a Method for Doing High-School Mathematics Carefully. Tech. Rep. 246, TUCS - Turku Centre for Computer Science.

Dijkstra, E., Scholten, C., 1990. Predicate Calculus and Program Semantics. Texts and Monographs in Computer Science. Springer-Verlag, New York.

Epp, S., 2003. The Role of Logic in Teaching Proof. American Mathematical Monthly (10), 886–899.

Gries, D., Schneider, F., 1993. A logical Approach to Discrete Math. Texts and Monographs in Computer Science. Springer-Verlag, New York.

Gries, D., Schneider, F., 1995. Teaching Math More Effectively Through the Design of Calculational Proofs. American Mathematical Monthly , 691–697.

Kavander, T., Nylund, M., Peltomäki, M., Salakoski, T., von Wright, J., 2001. Teaching High-School Mathematics with Structured Derivations in Hypertext Format. In: Proceedings of Kolin kolistelut – Koli Calling. The First Annual Finnish/Baltic Sea Conference on Computer Science Education. Report series A. University of Joensuu. Department of Computer Science, pp. 46–52.

Opetushallitus, 1994. Lukion opetussuunnitelman perusteet. (Principles of the High School Curriculum, in Finnish). Valtion painatuskeskus, Helsinki.

Opetushallitus, 2003. Lukion opetussuunnitelman perusteet. (Principles of the High School Curriculum, in Finnish). Valtion painatuskeskus, Helsinki.

Suikki, M., 2004. Arvosanojen korrelaatiotutkimusta. (Correlation studies on comprehensive and high-school grades). Dimensio (4), 16–21.

van Gasteren, A., 1990. On the Shape of Mathematical Arguments. Lecture Notes in Computer Science. Springer-Verlag, Berlin.